INDIAN STATISTICAL INSTITUTE
CHENNAI CENTRE
M.STAT I. 2014-15 Semester II
Multivariate Analysis
Mid-Semester Examination

Date : 02.03.2015 **Time : 3 hours**

This paper carries 108 marks. Answer as much as you can. Maximum you can score is 100. Marks are in [ ] and the end of each question. Do state the Theorems you are using in deriving your answers.

1. Let $\mathbf{X}_{n\times p}$ be a data matrix from a p-variate $\mathbf{x} \sim N_p(\mu, \Sigma)$, and $P_{n\times n}$ be an orthogonal matrix with the last row $\frac{1}{\sqrt{n}}$ times the unit vector. Suppose $\mathbf{Y} = P.\mathbf{X}$. Denote the rows of $\mathbf{Y}$ by $\mathbf{Y}_1, \mathbf{Y}_2, \ldots \mathbf{Y}_n$. Show that
   (a) $\mathbf{Y}_i$, $i \in \{1, 2, \ldots n\}$ are independent.
   (b) $\mathbf{Y}_n$ follows $N_p(\sqrt{n}\mu, \Sigma)$
   (c) $\mathbf{Y}_i$, $i \in \{1, 2, \ldots n-1\}$ are i.i.d. $N_p(0, \Sigma)$
   (d) Express $\bar{\mathbf{x}}$ in terms of $\mathbf{Y}_n$ only, and $\mathbf{A}$, the sum of square & sum of products matrix of the p-variate $\mathbf{x}$ in terms of $\mathbf{Y}_i$, $i \in \{1, 2, \ldots n-1\}$
   (e) State the distributions of $\bar{\mathbf{x}}$ and A, and comment on their dependence.[4+4+6+5+6=25]

2. Let $\mathbf{x} = (x_9, x_{10}, x_{11})'$ be a vector of weekly weights of a chicken for three consecutive weeks, namely 9^{th}, 10^{th} and 11^{th} Suppose $\mathbf{x} \sim N_3(\mu, \Sigma)$ where $\mu = (5, 4, 3)'$ and $\Sigma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ What is the distribution of the two weekly weight gains given the initial first week's weight or equivalently, what is the distribution of $(x_{10} - x_9, x_{11} - x_{10})'$ given x_9. Are $(x_{10} - x_9, x_{11} - x_{10})'$ and x_9 independently distributed? [8+7=15]

3. (a) Let $\mathbf{x} \sim N_n(\mu, \mathbf{I})$ and $Q_j = \mathbf{x}'A_j\mathbf{x}$, $j = 1, 2, \ldots, k$ where $\sum_{j=1}^{k} A_j = \mathbf{I}$. Then prove that $Q_1, Q_2, \ldots, Q_k$ are independently distributed as noncentral chi-squares with noncentrality parameters $\mu' A_j \mu$, $j = 1, 2, \ldots, k$ iff $\sum_{j=1}^{k} rank(A_j) = n$, or alternatively iff $A_1, , A_2, \ldots, A_k$ are idempotent matrices such that $A_iA_j = \mathbf{0}$ whenever $i \neq j$. Hence or otherwise answer the following questions.
   (b) i. Find Distributions of $q_1 = \mathbf{x}'A\mathbf{x}$, $q_2 = \mathbf{x}'B\mathbf{x}$
   ii. Show that q_1 and q_2 are independently distributed.
   Given that $\mathbf{x} \sim N_2(\mathbf{0}, \mathbf{I})$ and $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(c) Let $\mathbf{x} \sim N_n(\theta\mathbf{1}, \mathbf{I})$ and $\bar{x} = n^{-1}\mathbf{1}'\mathbf{x}$. Define A, idempotent such that

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \mathbf{x}'A\mathbf{x} = \mathbf{Y} \text{ (say)}$$

and hence prove that $\mathbf{Y}$ has a central χ^2 distribution with $(n-1)$ degrees of freedom, independent of that of $\bar{x}$. What would change in case $\mathbf{x} \sim N_n(\mu, \mathbf{I})$? [14+6+5=25]

4. Consider the following random variables $X_1 \sim N_1(0, 1)$, and

$$X_2|(X_1 = x_1) \sim N_2\left(\begin{pmatrix} 2x_1 \\ x_1 + 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Find the distribution of $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and hence or otherwise, the marginal of X_2.

[10+3=13]

5. Find the mean vector and the dispersion matrix of the random vector $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ with joint p.d.f. [5+10=15]

$$f(x_1, x_2) = \frac{1}{2\pi} exp\left[-\frac{1}{2}\{2x_1^2 + 5x_2^2 - 6x_1x_2 - 54x_1 + 84x_2 - 369\}\right]$$

6. Write short notes on

(a) Hoteling's T^2 statistic

(b) Likelihood Ratio Test for testing $H_0 \,:\, \mu = \mu_0$

(c) different types of Confidence regions and interval for a multivariate mean μ and it's components. [5×3=15]